

STUDY MATHS

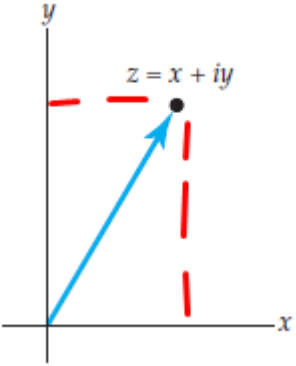
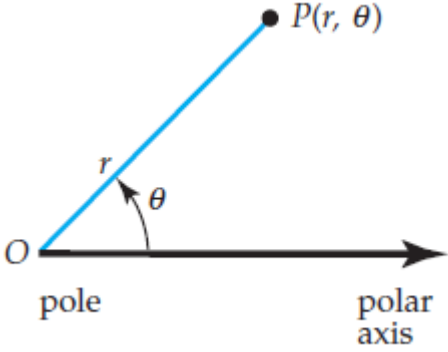
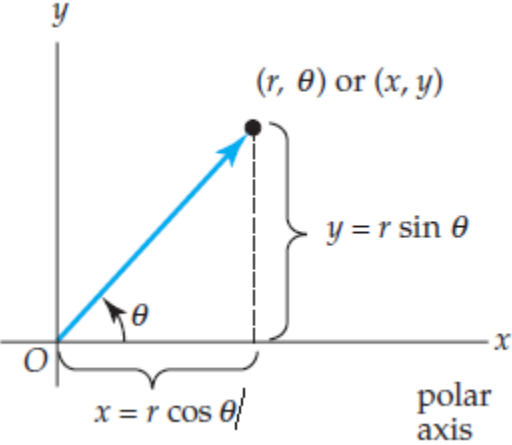
Complex Numbers

Heads Up:

$$\sqrt{-1} = i \rightarrow i^2 = -1$$

$$e^{i\frac{\pi}{2}} = i ; e^{-i\frac{\pi}{2}} = -i ; e^{i\pi} = -1 ; e^{i2\pi} = 1$$

$$Z = x + iy$$

Cartesian Form	Polar Form
	
$Z = x + iy$ $Re(Z) = x ; Img(Z) = y$	$Z = r e^{i\theta}$ $r = \sqrt{x^2 + y^2} = z $ $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$
$x = r \cos \theta ; y = r \sin \theta$	
	

Properties

Properties of modulus	Properties of conjugate
$ z = 0 \Rightarrow x = 0 ; y = 0$ $ z = \bar{z} = -z $ $ z_1 z_2 = z_1 z_2 $ $ z_1 + z_2 \leq z_1 + z_2 $ $ z_1 - z_2 \geq z_1 - z_2 $ $\left \frac{z_1}{z_2} \right = \frac{ z_1 }{ z_2 }$ $ z^n = z ^n$ $ z ^2 = (z \bar{z})$ $\left \int f(z) dz \right \leq \int f(z) dz $	$\bar{\bar{z}} = z - iy = r e^{-i\theta}$ $(\overline{z_1 + z_2}) = \bar{z}_1 + \bar{z}_2$ $(\overline{z_1 - z_2}) = \bar{z}_1 - \bar{z}_2$ $\overline{\left(\frac{z_1}{z_2} \right)} = \frac{\bar{z}_1}{\bar{z}_2}$ $z \bar{z} = x^2 + y^2 = z ^2$ $\overline{z^n} = \bar{z}^n$
Argument Properties	
$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$ $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ $\arg(z^n) = n \arg(z)$ $\arg(\bar{z}) = -\arg(z)$ $\arg\left(\frac{1}{z}\right) = -\arg(z)$	

$$z = x + iy ; \theta = \tan^{-1} \left(\frac{y}{x} \right) + 2n\pi \quad n = 0 \pm 1 \pm 2 \dots$$

$$\theta \text{ principle value} \quad -\pi \leq \theta \leq \pi$$

Demoiver Theorem

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Roots of Complex Num

$$Z^{\frac{1}{n}} = r^{\frac{1}{n}} \left[\cos \left(\frac{\theta + 2\pi k}{n} \right) + i \sin \left(\frac{\theta + 2\pi k}{n} \right) \right]$$

$$k = 0, 1, \dots, n-1$$

$$Z^{\frac{p}{q}} = r^{\frac{p}{q}} \left[\cos \left(\left(\frac{P(\theta + 2\pi k)}{q} \right) \right) + i \sin \left(\left(\frac{P(\theta + 2\pi k)}{q} \right) \right) \right]$$

$$k = 0, \dots, q-1$$

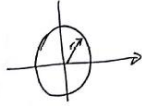
Functions of complex variables

$$W = F(z) = u(x, y) + i v(x, y)$$

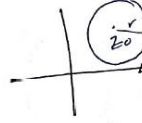
$$\text{let } W = F(z) = u + iv = \rho e^{i\phi}$$

* Some Curves & Region in Complex plane:

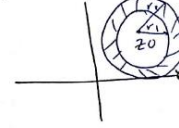
$$① |z| = r$$



$$* |z - z_0| = r$$



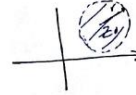
$$* r_1 \leq |z - z_0| \leq r_2$$



$$* |z| < r$$



$$* |z - z_0| < r$$



$$* |z - z_1| = k |z - z_2|$$

$$\text{if } k < 1$$

$$|z - z_1| = |z - z_2|$$



$$f(z) = e^z = e^{x+iy} = e^x (\cos y + i \sin y) \therefore u = e^x \cos y ; v = e^x \sin y$$

$$f(z) = \ln(z) = \ln r e^{i\theta} = \ln r + i(\theta + 2\pi k) \therefore u = \ln r = \ln \sqrt{x^2 + y^2} ; v = \theta = \tan^{-1} \frac{y}{x} + 2\pi k$$

Note :

في مسائل يعطيك علاقته $W=f(z)$ ثم يعطيك فترة الـ Z

الحل :

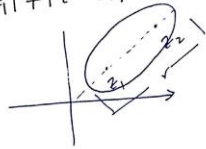
1- نرسم الفترة بتناح الـ Z بغض النظر عن العلاقة

2- على حسب العلاقة $f(z)$ نحدد شكل الـ W هتكون ازاوي

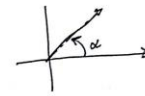
3- نرسم الـ W بتكون شبه رسمه الـ Z الاولى لكن بعد تاثير العلاقة فيها

$$* \text{if } k \neq 1 \Rightarrow \text{Circle convert to } z = x+iy \text{ Real.} \quad (5)$$

$$* |z - z_1| + |z - z_2| = r \Rightarrow \text{ellipse.}$$



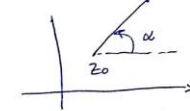
$$* \arg(z) = \alpha$$



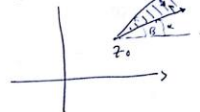
$$* |z - z_1| - |z - z_2| = r \Rightarrow \text{hyper.}$$



$$* \arg(z - z_0) = \alpha$$



$$* \arg(z - z_0) \leq \beta$$



Limit of Complex Fun.

$$\lim_{z \rightarrow z_0} f(z)$$

1- تعويض مباشر لو اتحلت خير وبركة limit exists لكن لو طلعت

$$-2 \quad \frac{\infty}{\infty} \text{ or } \frac{1}{0}$$

3- لو لقيت حاجات خاصة بال Complex زي $|z|, \bar{z}, \operatorname{Re}(z), \operatorname{Im}(z)$ نستخدم طريقة المسارات :

طريقة المسارات :

$$-1 \quad f(z) \rightarrow f(x, y) \text{ نحولها بدلاله } x, y$$

$$-2 \quad \text{put } y = y_0 \text{ then } \lim_{x \rightarrow x_0} f(x, y)$$

$$-3 \quad \text{put } x = x_0 \text{ then } \lim_{(y \rightarrow y_0)} f(x, y)$$

4- لو نتيجة الخطوتين 2 و 3 متساوية يبقى النهاية موجودة ، لو مش متساوية يبقى مش موجودة

Continuity of Complex fun.

Function is continuous if

$$1- f(z_0) \text{ exists} \quad 2- \lim_{z \rightarrow z_0} f(z) \text{ exist} \quad 3- (1) = (2) \text{ then the fun. is cont. at } z_0$$

الدوال متعددة التعريف { ندرس الاتصال عند نقط الانفصال

بعض قواعد الاتصال:

$$1 - \text{Polynomial cont. for all } z$$

$$2 - \sin, \cos, \sinh, \cosh, e^z \text{ cont for all } z$$

$$3 - f(z) = \frac{P}{Q} \text{ cont. for all } z \text{ except } \{ Q(z) = 0 \}$$

$$4 - \sec z = \frac{1}{\cos z} \rightarrow \text{cont for all } z \text{ except } z = \pm \frac{n\pi}{2}$$

$$5 - \operatorname{cosec} z = \frac{1}{\sin z} \rightarrow \text{cont for all } z \text{ except } z = \pm n\pi$$

Derivative of Complex Num.

- 1- لو حاجات عادية زي $\sin z$ و $\cos z$ او $1/z$ نستخدم قواعد الاشتقاق العادية
 2- لو لقيت حاجات خاصة بال Complex زي $|Z|$, $\bar{Z}$, $Re(z)$, $Im(z)$ نستخدم الطريقة الاتية

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \rightarrow \text{must exist}$$

او نفكها بدلالة x و y ونطبق المسارات في حل النهائية

$$\lim_{\Delta x \rightarrow 0} \lim_{\Delta y \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

$$\lim_{\Delta y \rightarrow 0} \lim_{\Delta x \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Analytic Fun.

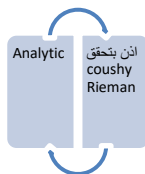
نقول على الداله انها اناليتك عند نقطة لو هي قابله للتفاضل عند النقطة والجوار بتاعها
 وعند دومين (مجال) لما تكون قابله للتفاضل عند كل نقاط الدومين

ونسميها entire عندما تكون قابله للتفاضل عند كل النقط في ال Complex Plane
 النقاط اللي تكون غير اناليتك عندها بس اناليتك حوالينها بنسميها singular points

Couchy Riemann equations

قوانين Couchy Rieman هي

Cartesian	Polar
$u_x = v_y$ $u_y = -v_x$	$u_r = \frac{v_\theta}{r}$ $v_r = -\frac{u_\theta}{r}$
$f'(z) = u_x + i v_x = v_y - i u_y$	$f'(z) = e^{-i\theta} [u_r + i v_r] = \frac{1}{r} e^{-i\theta} [v_\theta - i u_\theta]$



Harmonic Functions

Analytic $\Rightarrow$ Harmonic

Harmonic $\Rightarrow$ Achieve Laplace Equation ($\nabla^2 u = 0, \nabla^2 v = 0$)

Laplace equation :

$\nabla^2 u = 0$	$\nabla^2 v = 0$
$u_{xx} + u_{yy} = 0$	$v_{xx} + v_{yy} = 0$
$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$	$v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$

Notes

if $f = u + iv$ is harmonic solution of laplace $\therefore$ we call u harmonic conjugate of v

if $f = u + iv$ harmonic and if we draw $u = c_1$ and $v = c_2$ then their curves will be orthogonal

Elementary Functions

1- Exponential

$$e^z = e^{x+iy} = e^x [\cos(y + 2\pi k) + i \sin(y + 2\pi k)] \rightarrow \text{entire functions}$$

$$\begin{aligned} \text{Period} &: 2\pi i \\ \overline{e^z} &= e^{\bar{z}} \end{aligned}$$

Multi – one fun.

to make it one to one put $(-\pi < \theta \leq \pi)$ Principal Value

2- Complex Logarithm

$$\ln z = \ln |z| + i(\theta + 2\pi k)$$

Principal value : $\ln(z) = \ln r + i\theta$ $-\pi < \theta \leq \pi$ called principal branch

Analytic for all z except $\text{img}(z) = 0$ and $\text{Re}(z) \leq 0$ [$-ve x - axis$]

3- Complex Power

α : complex number

$$* Z^\alpha = e^{\alpha \ln z} = e^{\alpha [\ln r + i(\theta + 2\pi k)]}$$

Principal Value : $Z^\alpha = e^{\alpha [\ln r + i\theta]}$ where $-\pi < \theta \leq \pi$

$$\frac{d}{dz} z^\alpha = \alpha z^{\alpha-1} \text{ diff } \forall z \text{ except } (\text{Re}(z) \leq 0 \text{ \& } \text{img} = 0)$$

$$** \alpha^z = e^{z \ln \alpha}$$

$$\frac{d}{dz} \alpha^z = \alpha^z * \ln \alpha \text{ diff } \forall z$$

Trigonometric, hyperbolic functions

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \cos z = \frac{e^{iz} + e^{-iz}}{2} \rightarrow \text{entire of period } 2\pi$$

$$\tan z = \frac{\sin z}{\cos z}, \sec z = \frac{1}{\cos z} \rightarrow \text{entire except } z = \pm(2n+1)\frac{\pi}{2}$$

$$\operatorname{cosec} z = \frac{1}{\sin z}, \cot z = \frac{\cos z}{\sin z} \rightarrow \text{entire except } z = \pm n\pi$$

$$\text{Period : } \sin, \cos \rightarrow 2\pi$$

$$\text{Period : } \tan, \cot \rightarrow \pi$$

IDENTITIES :

$$\cos(-z) = \cos(z) \quad ; \quad \sin(-z) = -\sin z$$

$$\sin^2 z + \cos^2 z = 1$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$$

$$\sin 2z = 2 \sin z \cos z$$

$$\cos 2z = \cos^2 z - \sin^2 z$$

$$|\sin z| = \sqrt{\sin^2 x + \sinh^2 y}$$

$$|\cos x| = \sqrt{\cos^2 x + \sinh^2 y}$$

Hyperbolic

$$\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2} \rightarrow \text{entire period } 2\pi$$

$$\sinh(iz) = i \sin z \quad ; \quad \sin(iz) = i \sinh z$$

$$\cos iz = \cosh z \quad ; \quad \cosh iz = \cos z$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i \sinh(z)}{\cosh(z)} = i \tanh z$$

$$\sinh(-z) = -\sinh(z)$$

$$\cosh(-z) = \cosh(z)$$

$$\cosh^2 z - \sinh^2 z = 1$$

$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$$

$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$$

inverse

$$\sinh^{-1} z = \ln \left(z + \sqrt{z^2 + 1} \right)$$

$$\cosh^{-1} z = \ln \left(z + \sqrt{z^2 - 1} \right)$$

$$\tanh^{-1} z = \frac{1}{2} \ln \left(\frac{1+z}{1-z} \right)$$

$$\sin^{-1} z = -i \ln \left[iz + (1 - z^2)^{1/2} \right] \quad \cos^{-1} z = -i \ln \left[z + i(1 - z^2)^{1/2} \right]$$

$$\tan^{-1} z = \frac{i}{2} \ln \left(\frac{i+z}{i-z} \right) = \frac{1}{2i} \ln \left(\frac{1+iz}{1-iz} \right)$$

COMPLEX INTEGRALS

في التكامل تكامل على مسار نسميه $\int_C f(z) dz \Rightarrow F(z)$ Contour

لحلها : نفك ال $z = x + iy$ ثم من معادلة ال Contour نخليها تكامل في متغير واحد ونحلها عادي

لذا يكمل الحل على اكثر من صورة

$C : y = f(x)$ $\int_{x_1}^{x_2} f(x) dx$	$C : x = f(y)$ $\int_{y_1}^{y_2} f(y) dy$	$x = f(t) ; y = g(t)$ $\int_{t_1}^{t_2} f(t) dt$	$z = r e^{i\theta}$ $\rightarrow$ Polar for $\int_{\theta_1}^{\theta_2} f(\theta) d\theta$

Cauchy Theorems (closed C)

1- If fun. Is analtric in SCD (without holes) $\forall$ SCC in F $\oint_C f(z)dz = 0$

2- If fun. Is anlalytic in MCD (with holes) we use deformation إعادة تشكيل of contours

$$\oint_C = \oint_{C_1}$$

And generalization

$$\oint_C = \oint_{C_1} + \oint_{C_2} + \oint_{C_3} + \dots$$



3- When $\oint_C \frac{P(z)}{(z-z_0)^{n+1}} dz$ is not analytic in

C but $p(z)$ analytic in C

$Z_0 \in C$

$n = 0$	$n > 0$
$\oint \frac{p(z)}{z - z_0} dz$ $= 2\pi i p(z_0)$	$\oint_C \frac{P(z)}{(z - z_0)^{n+1}} dz$ $\frac{2\pi i}{n!} [P^n(z)]_{z=z_0}$

4- If fun. Analytic in SCD , C is any open contour $\therefore \int_C f(z) dz$ is independent on C يعني خذ أي مسار يعجبك

Bounding Theorem

If $f(z)$ is continuous on smooth curve C and $|f(z)| \leq M \forall z \in C$ then

$$\left| \int_C f(z) dz \right| \leq ML \text{ where } L \text{ is the lenght of } C$$

$$L = \int_a^b \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt$$



Maximum Modulus theorem

If $f(z)$ is continuous and analytic on closed curve C then

$|f(z)|$ attains the maximum on C

Cauchy Inequality

$$|f^n(z_0)| \leq \frac{n! M}{\rho^n}$$

Liouville's theorem

$$n = 1$$

*if it's analytic on the complex plane but bounded then it's **constant***

$$|f'(z_0)| \leq \frac{M}{\rho} \rightarrow 0$$

م الآخر لو الدالة ثابتة يبقى هي محدوده داخل الدومين على الحدود ، لكن لو كانت الدالة متغير اذن مستحيل تكون محدوده في الدومين ولكن اذا وجد C فان اقصى قيمة max boundary بتاعتها تكون على C نفسه

Taylor Series

If $f(z)$ is analytic on Domain and $z_0 \in D$ and $f(z)$ is analytic at z_0 then the series

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{f^k(z_0)}{k!} \right) (z - z_0)^k \rightarrow |z - z_0| < R \text{ (to be convergent)}$$

if $z_0 = 0$ it's Maclaurin series

$$f(z) = \sum_{k=0}^{\infty} \left(\frac{f^k(0)}{k!} \right) (z)^k \rightarrow |z| < R$$

وفي الاتي بعض ال series المشهورة والتي بنحفظها

Note : تفاضل وتكامل الدالة لا تعتبر من فترة التقارب :

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$ $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$ $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$ $\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$ $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$	$ z < \infty$
$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots$ $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots$ $(1+z)^n = 1 + nz + \frac{n(n-1)}{2!} z^2 + \frac{n(n-1)(n-2)}{3!} z^3 + \dots$	$ z < 1$

Radius of convergence $|z_0 - z^*|$

where z^* : nearest isolated singular point of the f in the D

Laurent Series

z_0 isolated singular point of f

$f(z) = \text{Principal Part} + \text{Analytic Part}$

$$= \dots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

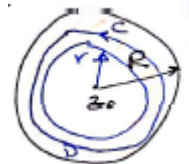
$$= \sum_{k=1}^{\infty} a_{-k} (z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

$$r < |z - z_0| < R$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \text{ cgt } r < |z - z_0| < R$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(s)}{(s-z_0)^{n+1}} ds$$



Zeros of $f(z)$

z_0 is a zero if $f(z_0) = 0$

z_0 is a zero of order m if $f(z_0) = f'(z_0) = f''(z_0) = \dots = f^{m-1}(z_0) = 0$

But $f^m(z_0) \neq 0$

Classification of Singularity

Singularity is a value that make $f(z_0) = \infty$

$$\text{Res} (f(z), z_0) = a_{-1} \quad 0 < |z - z_0| < R$$

$$\text{Res} = \frac{1}{z - z_0} \text{ معامل}$$

$$\oint_c f(z) dz = 2\pi i \sum \text{Res}$$

(1) Removable

if Principal part = 0

$$\lim_{z \rightarrow z_0} f(z) \neq \infty$$

$$\text{Res} = 0$$

(2) Pole

Simple	$\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$ $\therefore \text{Res} = \lim_{z \rightarrow z_0} [(z - z_0)f(z)]$ $\text{or } \text{Res} = \frac{f(z_0)}{g'(z_0)}$
Pole of order n	$= \frac{a_{-n}}{(z - z_0)^n} + \dots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0$ $+ a_1(z - z_0) + a_2(z - z_0)^2 + \dots$ $\text{Res} = \frac{1}{(n-1)!} \lim_{z \rightarrow z_0} \frac{d^{n-1}}{dz^{n-1}} [(z - z_0)^n f(z)]$

(3) Essentially

كل الحدود سالبة الاس

on; y principal part

$$\text{Ex} : \sin \frac{1}{z}, \cos \frac{1}{z}, \sinh \frac{1}{z}, \dots$$

مفاكيك ال z في المقام بصفى عامة

$$\lim_{z \rightarrow z_0} f(z) = \infty$$

$$\text{Res}(f(z), z_0) = \text{co. of } \left(\frac{1}{z - z_0} \right) = a_{-1} \rightarrow \text{from Laurent}$$

NOTE :

Before get Integral make sure that all res is in Contour !!!

اذن كده عندنا طريقتين لحل التكامل المركب

1- Cochy theorems 2- Res

جرب بكوشي منفعش جرب بال Res

Real Integral

Evaluate Real integral.

Handwritten notes for evaluating real integrals using complex analysis:

- Left Column (Unit Circle):**
 - $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$
 - $\oint_C f(z) dz$ with $C: |z|=1$
 - $z = e^{i\theta}, dz = i e^{i\theta} d\theta$
 - $\cos n\theta = \frac{z^n + 1/z^n}{2}$
 - $\sin n\theta = \frac{z^n - 1/z^n}{2i}$
 - $\oint_C = 2\pi i \sum \text{Res}$ (all poles inside)
- Middle Column (Rational Functions):**
 - $\int_{-\infty}^{\infty} f(x) dx$ where $f(x) = \frac{P(x)}{Q(x)}$
 - $P, Q \rightarrow \text{poly.}$
 - degree of $Q > P$ by 2
 - $\oint_C f(z) dz = \int_{-R}^R f(x) dx + \int_{CR} f(z) dz$
 - $\int_{-R}^R f(x) dx \xrightarrow{R \rightarrow \infty} \int_{-\infty}^{\infty} f(x) dx$
 - $\oint_C = 2\pi i \sum \text{Res}$ (poles inside contour)
- Right Column (Trigonometric Integrals):**
 - $\int_{-\infty}^{\infty} f(x) \cos nx$ or $\sin nx$
 - $f(x) = \frac{P(x)}{Q(x)}$ (poly)
 - $\oint_C f(z) \cdot e^{inz} dz$ (upper half plane for $n > 0$)
 - $\oint_C = \int_{-R}^R f(x) \cos nx dx + \int_{CR} f(z) e^{inz} dz$
 - $\int_{-R}^R f(x) \cos nx dx \xrightarrow{R \rightarrow \infty} \int_{-\infty}^{\infty} f(x) \cos nx dx$
 - $\oint_C = 2\pi i \sum \text{Res}$ (poles inside contour)

Diagrams show contours in the complex plane: a unit circle, a large semicircle in the upper half-plane, and a large semicircle in the lower half-plane.

Complex Inversion Formula

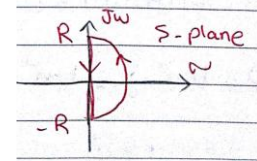
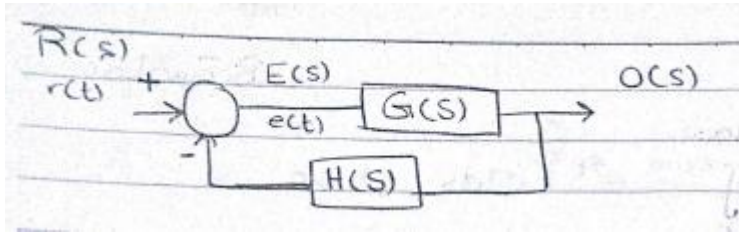
$$f(t) = L^{-1}[F(S)] = \frac{1}{2\pi i} \int_{\lambda+it}^{\lambda-it} e^{st} F(S) ds \quad t > 0$$

$$f(t) = 0 \quad t < 0$$

or (easier form)

$$f(t) = \sum \text{Res} [e^{st} F(S)]$$

Nyquist Diagram and Principle of argument



$$\text{Transfer function : } \frac{O}{R} = \frac{G(S)}{1 + GH \{ch. c eqn\}}$$

$$\phi(s) = S^3 + S^2 \rightarrow F(s) = \frac{S^2 + 1}{S^3 + S^2}$$

1- نمشي بداله ال فاي على المحور نضع $S = j\omega$

2- نجمع الحقيقي u والتخيلي v ونوجد $\phi = \tan^{-1} \frac{v}{u} (\omega \rightarrow \infty)$

3- نوجد $\phi_{net} = \phi_{end} - \phi_{beg}$

4- نمشي بداله الفاي على نص الدايره $S = Re^{i\theta}$

5- ناخذ الجزء اللي له تاثير S^3

6- نوجد معيار وزاويه $\rho = R^3 ; \phi = 3\theta$

7- من حدود ثيتا نوجد حدود فاي ونوجد نيت

8- $net \text{ variations of argw on } C = net + net$

9- $number \text{ of zeros} = \frac{1}{2\pi} (nvac)$

System of Linear ODE

1- Homogenous System

$$\dot{x} = A x, x(0) = x_0$$

$$\text{solution : } x(t) = e^{At} x_0 = (T e^{D_{\lambda} t} T^{-1}) x_0$$

1 – get λ :

$$\text{From } |A - \lambda I| = 0$$

2 – solve for x solutions

$$(A - \lambda I)x = 0 \text{ with gauss elimination}$$

3 – get T

$$T = (x_1 \ x_2 \ x_3 \dots)$$

4 – get $x(t)$

$$x(t) = (T e^{D_{\lambda} t} T^{-1}) x_0$$

$$x(t) = (x_1 \ x_2 \ x_3 \dots) \begin{pmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & e^{\lambda_2} & 0 \\ 0 & 0 & e^{\lambda_3} \end{pmatrix} (x_1 \ x_2 \ x_3 \dots)^{-1} x_0$$

2- Non-Homogenous system

$$\dot{x} = Ax + u(t), x(0) = x_0$$

$$\text{solution : } x(t) = e^{At} x_0 + e^{At} \int_0^t e^{-At} u(t) dt$$

Stability of D.E

$$\lambda = \alpha + i \beta$$

$\text{all } \alpha (-ve)$	Stable
$\alpha (-ve \text{ or } 0)$	Neutrally stable (not repeated)
One $\alpha (+ve)$	unstable

Series solution of ODE

$Q_2(x)y'' + Q_1(x)y' + Q_0(x)y = 0$ around x_0
put it on the form

$$y'' + \frac{Q_1}{Q_2} y' + \frac{Q_0}{Q_2} y = 0 \Rightarrow y'' + P y' + q y = 0$$

$\lim_{x \rightarrow x_0} P \neq \infty$ & $\lim_{x \rightarrow x_0} q \neq \infty$	$\lim_{x \rightarrow x_0} P = \infty$ or $\lim_{x \rightarrow x_0} q = \infty$		
$\therefore x_0$ is an ((ordinary point)) for D.E $y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ $y' = \sum_{n=1}^{\infty} a_n (n) (x - x_0)^{n-1}$ $y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) (x - x_0)^{n-2}$ $x_0 = 0$ $y = \sum_{n=0}^{\infty} a_n (x)^n$ $y' = \sum_{n=1}^{\infty} a_n (n) (x)^{n-1}$ $y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) (x)^{n-2}$	$\therefore x_0$ is a ((Singular Point)) for D.E <table border="1" style="width: 100%; border-collapse: collapse;"> <tr> <td data-bbox="597 741 1193 1654" style="width: 50%; padding: 10px;"> $\lim_{x \rightarrow x_0} (x - x_0)^P \neq \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q \neq \infty$ Regular s.p $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x - x_0)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x - x_0)^{n+\lambda-2}$ Put $(x - x_0) = \omega$ and substitute in D.E $x_0 = 0$ $y = \sum_{n=0}^{\infty} a_n (x)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x)^{n+\lambda-2}$ </td><td data-bbox="1193 741 1526 1654" style="width: 50%; padding: 10px;"> $\lim_{x \rightarrow x_0} (x - x_0)^P = \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q = \infty$ Irregular s.p </td></tr> </table>	$\lim_{x \rightarrow x_0} (x - x_0)^P \neq \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q \neq \infty$ Regular s.p $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x - x_0)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x - x_0)^{n+\lambda-2}$ Put $(x - x_0) = \omega$ and substitute in D.E $x_0 = 0$ $y = \sum_{n=0}^{\infty} a_n (x)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x)^{n+\lambda-2}$	$\lim_{x \rightarrow x_0} (x - x_0)^P = \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q = \infty$ Irregular s.p
$\lim_{x \rightarrow x_0} (x - x_0)^P \neq \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q \neq \infty$ Regular s.p $y = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x - x_0)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x - x_0)^{n+\lambda-2}$ Put $(x - x_0) = \omega$ and substitute in D.E $x_0 = 0$ $y = \sum_{n=0}^{\infty} a_n (x)^{n+\lambda}$ $y' = \sum_{n=0}^{\infty} a_n (n + \lambda) (x)^{n+\lambda-1}$ $y'' = \sum_{n=0}^{\infty} a_n (n + \lambda)(n + \lambda - 1) (x)^{n+\lambda-2}$	$\lim_{x \rightarrow x_0} (x - x_0)^P = \infty$ or $\lim_{x \rightarrow x_0} (x - x_0)^2 q = \infty$ Irregular s.p		

Solution Steps : (Ordinary Point)

- 1- Decide the type of the point
- 2- Substitute the new $y' y''$ relation in the equation
- 3- Equate the least power co. eff. = 0

Solution steps (singular point)

- 1- Substitute in D.E
- 2- Equate the least power co. eff. to get R.R
- 3- Put $(n=0)$ in RR the $a_n = a_0$ to get an equation in λ
- 4- Solution λ_1, λ_2

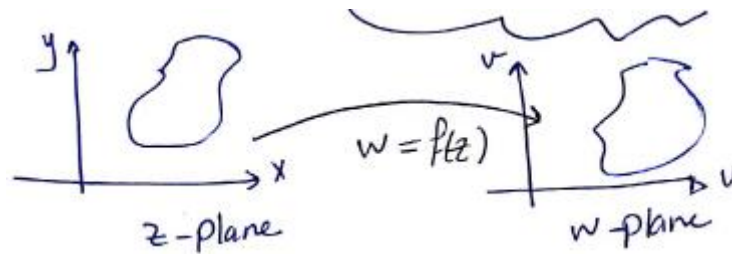
$\lambda_1 - \lambda_2 \neq \text{integer}$	$\lambda_1 = \lambda_2$	$\lambda_1 - \lambda_2 = \text{integer}(k)$	
$y(x) = c_1 y_1(x) + c_2 y_2(x)$ $y_1(x) = y(x, \lambda_1)$ $\quad = x^{\lambda_1} \sum a_n x^n$ $y_2(x) = y(x, \lambda_2)$ $\quad = x^{\lambda_2} \sum a_n x^n$	$y_1 = y(x, \lambda_1)$ $y_2 = \left[\frac{\partial(y(x, \lambda))}{\partial \lambda} \right]_{\lambda=\lambda_2}$ Or $y_2 = y_1 \int \frac{(e^{-\int p(x) dx})}{y_1^2} dx$	$[a_k]_{\lambda=\lambda_2} = \frac{0}{0}$	$[a_k]_{\lambda=\lambda_2} = \infty$
		$y(x) = y(x, \lambda_2)$	<i>replace</i> $a_0 = c(\lambda - \lambda_2)$ Obtain $y(x, \lambda)$ $y_1(x) = y(x, \lambda_1)$ $y_2 = \left[\frac{\partial(y(x, \lambda))}{\partial \lambda} \right]_{\lambda=\lambda_2}$

at large $x \rightarrow \omega = \frac{1}{x}$

$$y' = \frac{dy}{d\omega} (-\omega^2) ; y'' = \omega^4 \frac{d^2 y}{d\omega^2} - 2\omega^3 \frac{dy}{d\omega}$$

then substitute when $\omega = 0$ s.p or ordinary

Mapping



Mapping types

Translation	Magnification(stretching)	Rotation
$f(z) = Z + b$ $= (x, y) + (x_0, y_0)$ $= (x + x_0) + i(y + y_0)$	$w = f(z) = aZ = ar e^{i\theta}$ $a > 1$ (real)	$w = BZ = r e^{i(\theta+\theta_1)}$ $ B = 1$ rotates by θ_1 $w = e^{i\theta} z$
		If $ B \neq 1$ its mag. & Rot.

Linear Mapping	BiLinear Mapping
$W = BZ + \alpha$ $B, \alpha \rightarrow \text{complex}$	$W = \frac{\alpha z + \beta}{\gamma z + \delta} \Rightarrow \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$
This will : 1- Rotate by angle of $\arg(B)$ 2- Magnification by $ B $ 3- Translate by α	This will : Convert upper half plane into a circle and vice versa and other apps.

خطوات حل مسائل زي دي

- 1- $W = f(z)$
- 2- نعوض ونفك
- 3- نقسم جزء حقيقي u و جزء تخيلي v
- 4- Find $v = f(u)$ then plot it

Note : $V^2 = 4C^2(C^2 - U)$ parabola open to left

Rate change between w and z

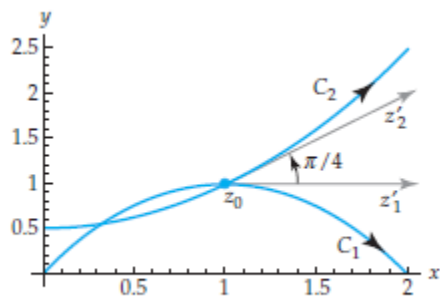
$$\frac{d\omega}{dt} = \frac{d\omega}{dz} * \frac{dz}{dt} \Rightarrow \Delta\omega = |f'(z)| \Delta z$$

$$dxdy = \left(\frac{\partial(u,v)}{\partial(x,y)} \right) dudv \Rightarrow dxdy = |f'(z)|^2 dudv$$

$$\phi_0 = \theta_0 + \arg(f'(z_0))$$

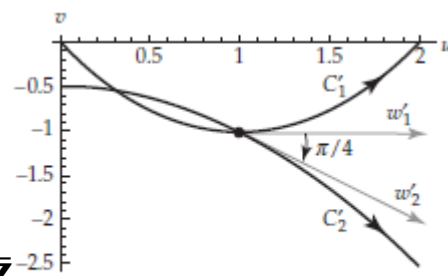
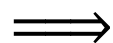
Conformal Mapping

Mapping is called conformal if θ between C_1 & $C_2 = \theta$ between C'_1 & C'_2 in magnitude and sense



(a) Curves C_1 and C_2 in the z -plane

$$W = \bar{Z}$$



(b) Images of the curves in (a) under $w = \bar{z}$

isogonal

NOTE : if f is analytic function in Domain containing Z_0 and $f'(z_0) \neq 0$ then $w = f(z)$ is called conformal mapping

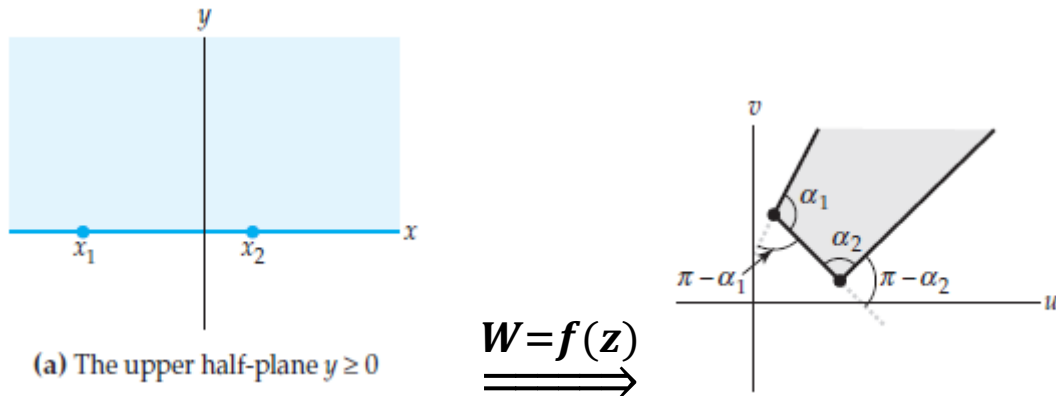
$$\text{Angle of Rotation} \Rightarrow \arg(f'(z_0))$$

$$\text{angle of rotation} \neq \theta$$

Mapping of upper half plane(UHP)

UHP to circle	$W = e^{i\theta} \frac{z - z_0}{z - \bar{z}_0} \quad z_0 \in UHP$
Circle to UHP	$Z = e^{i\theta} \frac{w - w_0}{w - \bar{w}_0}$

Schwarz- Christoffel transformation (polygonal)



converts $x_1, x_2, \dots, x_n \rightarrow w_1, w_2, w_3 \dots w_n$

To get $w = f(z)$

$$\frac{dw}{dz} = A(z - x_1)^{\frac{\alpha_1}{\pi} - 1} (z - x_2)^{\frac{\alpha_2}{\pi} - 1} \dots (z - x_n)^{\frac{\alpha_n}{\pi} - 1}$$

Solve laplace equation using mapping

Laplace eqn. $\nabla^2 \phi = 0$

$$\phi_{xx} + \phi_{yy} = 0 \xrightarrow{\text{by mapping}} \phi_{uu} + \phi_{vv} = 0$$

To sol for ϕ

We always solve laplace equation in **UHP** , if not convert to it

$$\phi = A_1 \theta_1 + A_2 \theta_2 + \dots + A_n \theta_n + B$$

Where : (elly gay dah malosh lazma)

$$\phi(x, 0) = \begin{cases} k_0, & -\infty < x < x_1 \\ k_1, & x_1 < x < x_2 \\ \vdots & \vdots \\ k_n, & x_n < x < \infty, \end{cases}$$

harmonics because it's img part of : $A_1 \ln(z - x_1) + A_2 \ln(z - x_2) + \dots + A_n \ln(z - x_n) + iB$

Special Functions

(1) Gamma Function

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \quad , n > 0$$

$$\Gamma(n+1) = n! \quad (n + ve \text{ integer}) \quad ex: \Gamma 5 = 4!$$

$$\Gamma(n+1) = n \Gamma n \quad (n + ve \text{ fraction}) \quad ex: \Gamma\left(\frac{3}{2}\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n} \quad (n - ve \text{ fraction}) \quad ex: \Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{-\frac{3}{2}}$$

special :

$$\Gamma 1 = \Gamma 2 = 1 \parallel \Gamma 0 = \Gamma(-1) = \Gamma(-ve) = \infty \parallel \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x} \rightarrow \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{\sin\left(\pi * \frac{1}{4}\right)} = \pi\sqrt{2}$$

Asymptotic formula for gamma (when n is very large)

$$\Gamma(n+1) = n! \cong \sqrt{2n\pi} n^n e^{-n} \quad (n \text{ integer})$$

(2) Beta Function

$$B(m, n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad m, n > 0$$

$$(x = \sin^2 \theta) \Rightarrow B(m, n) = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta$$

$$(x = 1 - y) \Rightarrow B(n, m) = \int_0^1 (1-y)^{m-1} y^{n-1} dy$$

$$\left(x = \frac{y}{a}\right) \Rightarrow B(m, n) = \frac{1}{a^{m+n-1}} \int_0^a y^{m-1} (a-y)^{n-1} dy$$

$$\left(x = \frac{y}{1+y}\right) \Rightarrow B(m, n) = \int_0^{\infty} \frac{(y^{m-1})}{(1+y)^{m+n}} dy$$

$$B(m, n) = B(n, m)$$

$$B(m, n) = \frac{m-1}{n+m-1} B(m-1, n)$$

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad (\text{used to get definite integral})$$

$$\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin m\pi} = B(m, 1-m) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$$

Special :

$$B(1,1) = 1 \quad - - \quad B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi$$

Steps of evaluating integral with beta

$$I = \int_0^{\frac{\pi}{2}} (\sin \theta)^P (\cos \theta)^q d\theta$$

$$\text{From def of B } \mathbf{B(m, n)} = 2 \int_0^{\frac{\pi}{2}} (\sin \theta)^{2m-1} (\cos \theta)^{2n-1} d\theta$$

$$P = 2m - 1 \therefore m = \frac{P+1}{2}$$

$$q = 2n - 1 \therefore n = \frac{q+1}{2}$$

$$\text{the value of } I = \frac{1}{2} B(m, n) = \frac{1}{2} B\left(\frac{P+1}{2}, \frac{q+1}{2}\right)$$

(3) Bessel Function

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad \text{near } x_0 = 0$$

It's general solution using frobenius method

$$y(x) = C_1 J_n(x) + C_2 Y_n(x)$$

Bessel fn
Order n
Kind 1

Bessel fn
Order n
Kind 2

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r! \Gamma(n+r+1)}$$

$$J'_0 = -J_1$$

Recurrence relations for bessel fun.

$$(1) J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

$$(2) J'_n(x) = \frac{1}{2} [J_{n-1}(x) - J_{n+1}(x)]$$

$$(3) x J'_n(x) = x J_{n-1}(x) - n J_n(x)$$

$$(4) \frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x)$$

$$(5) \frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$$

$$(6) J_{-n}(x) = (-1)^n J_n(x)$$

Handwritten note:
 $\Rightarrow \frac{d}{dx} [x J_1(x)] = x J_0(x)$
 $\int x J_0(x) dx = x J_1(x) + C$

Asymptotic Formula for Bessel function (x very large)

$$J_n(x) \cong \sqrt{\frac{2}{x\pi}} \cos\left(x - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

Bessel generating function

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Wrt x	Wrt t
$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} * \frac{1}{2}\left(t - \frac{1}{t}\right) = \sum_{n=-\infty}^{\infty} t^n J'_n(x)$	$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} * \frac{x}{2}\left(1 + \frac{1}{t^2}\right) = \sum_{n=-\infty}^{\infty} n t^{n-1} J_n(x)$